(3.10) $$\hbar\omega\left(\frac{1}{\sqrt{2}}\left(q-\frac{\partial}{\partial q}\right)\cdot\frac{1}{\sqrt{2}}\left(q+\frac{\partial}{\partial q}\right)\ +\frac{1}{2}\right)|\psi(q)\rangle \quad \doteq \quad E_\omega|\psi(q)\rangle$$

From these two innermost brackets, we now introduce the following ladder operators.

3.1.3. Ladder Operators of the Quantum Harmonic Oscillator

(3.11) $a \coloneqq \frac{1}{\sqrt{2}}(q+i\hat{p}) = \frac{1}{\sqrt{2}}\left(q+\frac{\partial}{\partial q}\right)$ annihilation operator (lowering operator)

(3.12) $a^\dagger \coloneqq \frac{1}{\sqrt{2}}(q-i\hat{p}) = \frac{1}{\sqrt{2}}\left(q-\frac{\partial}{\partial q}\right)$ creation operator (raising operator)[69]

From these two, we can also write the operators:
The real *field* ***quantity*** operator[70] $\hat{q}$ and the *complex field* ***momentum*** operator $\hat{p}$

(3.13) $\hat{q} = \frac{1}{\sqrt{2}}(a^\dagger + a)$ and $\hat{p} = i\frac{1}{\sqrt{2}}(a^\dagger - a)$ or $\frac{\partial}{\partial q} = \frac{1}{\sqrt{2}}(a - a^\dagger)$

We notice immediately the commutator between
the annihilation and the creation operators

(3.14) $[a, a^\dagger] = 1,$

because, $[a,a^\dagger] = \frac{1}{2}[(q+i\hat{p}),(q-i\hat{p})] = \frac{1}{2}([q,-i\hat{p}]+[i\hat{p},q]) = -i\frac{1}{2}([q,\hat{p}]+[q,\hat{p}]) = 1,$ according to (2.71).

From a and $a^\dagger$ the stationary Schrödinger eigenvalue equation (3.10) can be written

(3.15) $$\boxed{\hbar\omega\left(a^\dagger a+\frac{1}{2}\right)|\psi\rangle \quad \doteq \quad E_\omega|\psi\rangle}$$

As we see, the oscillating Hamilton operator $\hat{H}_\omega$ is always proportional to the given ***quantity*** ω, and it can due to the commutator (3.14) be expressed in two forms

(3.16) $\hat{H}_\omega = \hbar\omega\left(a^\dagger a+\frac{1}{2}\right)$

(3.17) $\hat{H}_\omega = \hbar\omega\left(a a^\dagger-\frac{1}{2}\right)$

These can be combined in a single expression by half of the canonical addition of these two

(3.18) $\hat{H}_\omega = \frac{\hbar\omega}{2}(a^\dagger a + a a^\dagger)$

We note that the following commutator applies in oscillating quantum mechanics[71]

(3.19) $[\hat{H}_\omega, a^\dagger] = \hbar\omega a^\dagger,$

as, $[\hat{H}_\omega,a^\dagger] = \left[\hbar\omega\left(a^\dagger a+\frac{1}{2}\right),a^\dagger\right] = \hbar\omega\left(a^\dagger a a^\dagger+\frac{1}{2}a^\dagger - a^\dagger a^\dagger a - a^\dagger\frac{1}{2}\right) = \hbar\omega a^\dagger[a,a^\dagger] = \hbar\omega a^\dagger$

(3.20) $[\hat{H}_\omega, a] = -\hbar\omega a\ .$

3.1.4. Eigenstates in the Real Field *Linear* Quantum Harmonic Oscillator

In line with the general eigenvalue equations (2.66) and (2.67) for the harmonic oscillator, we now write the eigenvalue equation for the quantum harmonic oscillator in the form

(3.21) $\hat{H}_\omega|\psi_n\rangle \doteq E_{\omega,n}\,|\psi_n\rangle$

We are now looking at a possible special eigenstate condition and let the annihilation operator a respectively the creation operator $a^\dagger$ work on this eigenstate and we get for $a^\dagger|\psi_n\rangle$ and $a\,|\psi_n\rangle$ using (3.19) and (3.20) the following eigenvalue equations

(3.22) $\hat{H}_\omega a^\dagger|\psi_n\rangle \doteq (E_{\omega,n}+\hbar\omega)a^\dagger|\psi_n\rangle$

(3.23) $\hat{H}_\omega a\,|\psi_n\rangle \doteq (E_{\omega,n}-\hbar\omega)\,a\,|\psi_n\rangle$

[69] Note that we leave out ^above $a^\dagger$ and a because they do not have classical analogies they must be distinguished from.
[70] Note that the operator $\hat{q} = q\,\hat{1}$ is the real ***field quantity*** q multiplied a given unit vector $\hat{1}$ as a ***direction quality*** (where in?).
[71] Compare also with (2.72), (2.75) and (2.76), and also (2.27) and (2.28) from the classic case.

Research on the a priori of Physics
Geometric Critique of Pure Mathematical Reasoning
Jens Erfurt Andresen
Edition 2, © 2020-22

We see that $a^\dagger|\psi_n\rangle$ and $a\,|\psi_n\rangle$ are eigenvectors with eigenvalues

(3.24) $(E_{\omega,n}+\hbar\omega)$ and $(E_{\omega,n}-\hbar\omega)$.

From this, we can also write $a^\dagger|\psi_n\rangle = |\psi_{n+1}\rangle$ and $a|\psi_n\rangle = |\psi_{n-1}\rangle$. This makes the operators $a^\dagger$ and a to the raising- respectively lowering-operator for these eigenstates.
The energy difference between two eigenstates $|\psi_n\rangle$ and $|\psi_{n\pm1}\rangle$ becomes $\Delta E = \pm\hbar\omega$.
For the eigenvalues, we can write $E_{\omega,n+1} = (E_{\omega,n}+\hbar\omega)$ and $E_{\omega,n-1} = (E_{\omega,n}-\hbar\omega)$.
We are defining the numbering of the eigenstates $|\psi_n\rangle$ from the natural numbers $n \in \mathbb{N}$.
Hence, we must ask for the ground state n=0. what value does $E_{\omega,0}$ for $|\psi_0\rangle$ have?
Because $0-1 \notin \mathbb{N}$, we will assume that $a|\psi_0\rangle$ disappears, therefore we set the value of

(3.25) $a|\psi_0\rangle = 0\cdot|\psi_0\rangle = 0\,.$

From $|\psi_0\rangle$ in (3.21) we obtained $\hat{H}_\omega|\psi_0\rangle = E_{\omega,0}\,|\psi_0\rangle$, and from (3.16), we get

(3.26) $\hbar\omega\left(a^\dagger a+\frac{1}{2}\right)|\psi_0\rangle = E_{\omega,0}\,|\psi_0\rangle.$

By pulling over $\frac{1}{2}\hbar\omega$, and turning the equation, as well as utilise $a|\psi_0\rangle$=0 we obtain

(3.27) $\left(E_{\omega,0}-\frac{\hbar\omega}{2}\right)|\psi_0\rangle = \hbar\omega\, a^\dagger a|\psi_0\rangle = \hbar\omega\, a^\dagger 0\cdot|\psi_0\rangle = 0\cdot|\psi_0\rangle \;\Rightarrow\; E_{\omega,0} = \frac{1}{2}\hbar\omega.$

The ground state $|\psi_0\rangle$ is not negligible and its eigenvalue is $E_{\omega,0} = \hbar\omega/2$, so

(3.28) $\hat{H}_\omega|\psi_0\rangle = \frac{1}{2}\hbar\omega\,|\psi_0\rangle$

The raised eigenvalues for the real field linear harmonic oscillator are successively given as

(3.29) $E_{\omega,n} = \left(n+\frac{1}{2}\right)\hbar\omega$

3.1.5. The Quantum Number Operator

From the creation operator acting on the annihilation operator, we form the number operator.

(3.30) $\hat{N} \coloneqq a^\dagger a$

$\hat{N}$ acting on the eigenstate reads the number of the state $|\psi_n\rangle$ as an eigenvalue n of $\hat{N}$.

(3.31) $\hat{N}|\psi_n\rangle = n\,|\psi_n\rangle$

We write the commutator relations

(3.32) $[\hat{N}, a^\dagger] = a^\dagger,\quad [\hat{N}, a] = -a\,,$ in that $[a, a^\dagger] = 1,$ according to (3.14).

We see from (3.16) that the Hamilton operator can be expressed by the number operator

(3.33) $\hat{H} = \left(\hat{N}+\frac{1}{2}\right)\hbar\omega$

Thus $[\hat{N},\hat{H}] = [\hat{H},\hat{N}] = 0$ and we see that eigenstates $|\psi_n\rangle$ for $\hat{H}$ are also eigenstates for $\hat{N}$.
From (3.32), in that $\hat{N}a^\dagger = a^\dagger\hat{N} + [\hat{N},a^\dagger] = a^\dagger\hat{N} + a^\dagger \sim (n+1)a^\dagger$, we find

(3.34) $\hat{N}a^\dagger\,|\psi_n\rangle = (n+1)a^\dagger|\psi_n\rangle$

(3.35) $\hat{N}a\,\,|\psi_n\rangle = (n-1)\,a\,|\psi_n\rangle$

Here we find the idea by the introducing of the creation operator $a^\dagger$ and the annihilation operator a frees us from the dependency on the frequency energy factor ***quantity*** $\hbar\omega$. In this way, we can operate from their eigenstates without including the energy or frequency ***spectrum*** and thus also omit the aspect of the canonical conjugated development parameter t. Causality by number operation $\hat{N}$ is now exclusively caused by the use of $a^\dagger$ after a, which is a re-creation-after an annihilation- operation, joined in one operation, as one count.