Research on the a priori of Physics
Geometric Critique of Pure Mathematical Reasoning
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3. The Quantum Harmonic Oscillator

We start with the Hamilton function (2.91) for the classical harmonic oscillator

(3.1) $$H_\omega(q_\omega, p_\omega) = \frac{p_\omega^2}{2\,m_\omega} + \frac{1}{2} m_\omega \omega^2 q_\omega^2$$

In quantum mechanics, we try, to convert it to an operator $\widehat{H}$ that can meet the stationary Schrödinger eigenvalue equation $\widehat{H}|\psi\rangle \doteq E_\omega |\psi\rangle$, (e.g. see. (2.64) to (2.67)).

3.1.2. The *Quantum Real Scalar Field* for the Linear Harmonic Oscillator

We transform (3.1)↔(2.91) to the operator edition, by first rewriting the ***quantity*** $q_\omega \in \mathbb{C}$ to a ***real field quantity*** $q \in \mathbb{R}$,[67] where we use Planck's constant $\hbar$, which is marked green.

(3.2) $$q \sim \sqrt{\frac{m\omega}{\hbar}} \cdot |q_\omega|, \quad \text{or quadratic as the last term of (3.1)} \quad \hbar\omega q^2 \equiv m\omega^2 |q_\omega|^2 = m\omega^2 (q_\omega^* q_\omega) \geq 0.$$

Then we rewrite the momentum p_ω from this normalised ***field quantity*** q using the *momentum operator* (2.81) in accordance with (2.71) $[q, \hat{p}] = i$

(3.3) $$\hat{p} = -i\frac{\partial}{\partial q} = -i\left(\frac{\partial q_\omega}{\partial q}\right)\frac{\partial}{\partial q_\omega} \overset{\hbar}{\sim} -i\frac{1}{\hbar}\sqrt{\frac{\hbar}{m\omega}} \cdot \frac{\partial}{\partial q_\omega} \backsim \sqrt{\frac{1}{\hbar m\omega}} \cdot p_\omega, \quad \text{or quadratic}$$

$$(p_\omega)^2 \sim -\hbar m\omega \frac{\partial^2}{\partial q^2}, \quad \text{because} \quad p_\omega \sim \sqrt{\hbar m\omega}\left(-i\frac{\partial}{\partial q}\right) = \sqrt{\frac{m\omega}{\hbar}}\left(-i\hbar\frac{\partial}{\partial q}\right)$$

By substitutions in $H_\omega(q,p)$ we rewrite the Hamilton operator $\widehat{H}_\omega$ for the harmonic oscillator

(3.4) $$H_\omega = \frac{p_\omega^2}{2\,m_\omega} + \frac{1}{2} m_\omega \omega^2 q_\omega^2 \sim \widehat{H}_\omega = \frac{\hbar m\omega}{2m}\frac{\partial^2}{\partial q^2} + \frac{\hbar m\omega^2}{2m\omega} q^2 = \frac{\hbar\omega}{2}\left(\hat{p}(\hat{p}) + q^2\right) = \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial q^2} + q^2\right).$$

We note that a local external m has no impact as it is not included in the last formulation.[68]
To prevent the classical Hamilton function loses its meaning, we can choose an internal impedance factor $m_\omega \neq 0$ by choosing $m_\omega c^2 \sim \hbar\omega$, $(c=1)$, imply that $q \sim \omega |q_\omega|$ and $p_\omega \sim \hbar\omega\hat{p}$. (refer Figure 1.3)
In this way, you can understand a cyclic oscillation in a quantum-mechanical field equivalent to a 'classic' harmonic oscillator. – I now propose a new interpretation:
We intuit $q \in \mathbb{R}$ as a purely *isotropic scalar field* with **no** specific ***direction*** in any space!
The Hamilton operator for the harmonic oscillator of the ***field quantity*** q is now

(3.5) $$\widehat{H}_\omega = \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial q^2} + q^2\right) = \frac{\hbar\omega}{2}\left(q^2 - \frac{\partial^2}{\partial q^2}\right)$$

Hence, we write the stationary Schrödinger eigenvalue equation as

(3.6) $$\frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial q^2} + q^2\right) |\psi(q)\rangle \doteq E_\omega |\psi(q)\rangle$$

The expression in parenthesis can be rewritten as

(3.7) $$q^2 - \frac{\partial^2}{\partial q^2} = \left(q - \frac{\partial}{\partial q}\right)\left(q + \frac{\partial}{\partial q}\right) + \frac{\partial}{\partial q} q - q\frac{\partial}{\partial q}$$

The last two terms constitute the commutator

(3.8) $$\left[\frac{\partial}{\partial q}, q\right] = 1, \quad \text{because} \quad \left[\frac{\partial}{\partial q}, q\right]\psi = \frac{\partial}{\partial q}(q\psi) - q\frac{\partial}{\partial q}\psi = q\frac{\partial}{\partial q}\psi + \frac{\partial q}{\partial q}(\psi) - q\frac{\partial}{\partial q}\psi = 1(\psi), \quad \text{therefore}$$

(3.9) $$q^2 - \frac{\partial^2}{\partial q^2} = \left(q - \frac{\partial}{\partial q}\right)\left(q + \frac{\partial}{\partial q}\right) + 1$$

Now the stationary Schrödinger eigenvalue equation (3.6) is written as

[67] A *field* is a commutative ring of numbers, e.g., scalars as real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, that therefore are called *scalar fields*. The reason for a real scalar field $q \in \mathbb{R}$ here is, that we first look at a *linear* harmonic oscillator. And omit the complex part $q \leftrightarrow \tilde{q}_\omega \in \mathbb{R}$ of $q_\omega(t) = \tilde{q}_\omega e^{-i\omega t} \in \mathbb{C}$. The blue quantity q indicate its origin from a ***quantity*** scalar field operator $\hat{q} = i\frac{\partial}{\partial p}$.
[68] This is the big shift that happens when you go from Classical Mechanics to modern ***quantum field theory***.

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