(2.55) $\{f,g\} \leftrightarrow -i[\hat{q},\hat{p}], \qquad \{f,g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \overset{i}{\longleftrightarrow} [\hat{q},\hat{p}] = \hat{q}\,\hat{p} - \hat{p}\,\hat{q}.$

Classical $\leftrightarrow$ Quantum, The corresponding association between the two mechanics.

(2.56) $\{f,H\} \leftrightarrow -i\left[\hat{F}(t),\hat{\omega}\right]$

Thus, we see that we can introduce a *Hamilton operator* $\hat{H}$ we have the relationship

(2.57) $\hat{H} \leftrightarrow \hat{\omega}$ or even[57] $\hat{H}_\omega = \hbar\hat{\omega}.$

We have that the frequency operator is synonymous with the *Hamilton operator*, which provides the quantum energy. Planck's constant $\hbar$ is a constant relationship between the angular frequency and the quantum energy.

You can choose $\hbar$=1, when the unit of measure for $\hat{H}\sim E_\omega$ and $\hat{\omega}\sim 2\pi f$ is the same. E.g.:

- Quantum energy E_ω can be measured in [(2π)Hz], which is the angular radian rotation [per second], or
- Angle frequency ω may be measured in [eV] or [Joule], which are units of energy.[58]

2.2.3.3. Schrödinger Picture

We look at an arbitrary stationary probability function $|\psi\rangle$ and compare it with ***quantities*** of the type $\tilde{q}_\omega$ from the circle oscillator.

(2.58) $|\psi_\omega\rangle \sim \tilde{q}_\omega$

The parameter dependent probability function (2.47) is compared with (2.42);

(2.59) $|\psi_\omega(t)\rangle = e^{-i\hat{\omega}t}|\psi_\omega\rangle \sim e^{-i\hat{\omega}t}\tilde{q}_\omega = e^{-i\hat{\omega}t}(\tilde{q}_\omega) \sim \varphi_\omega(t).$

By further comparing with (2.43) we get the possibility of an evolutionary probability function

(2.60) $|\psi(t)\rangle = \int_{\mathbb{R}} d\omega\, e^{-i\omega\cdot t}\,|\psi_\omega\rangle \sim q(t) = \int_{\mathbb{R}} d\omega\, e^{-i\omega\cdot t}\,\tilde{q}(\omega) \sim \varphi(t).$

By incorporating $|\psi(t)\rangle \sim \varphi(t)$ in (2.40) we get the equation

(2.61) $\boxed{i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{\omega}\,|\psi(t)\rangle}\,.$

Here we write the evolution of a probability ***quantity*** $|\psi(t)\rangle$ as a parameter-dependent complex oscillation operator $e^{-i\hat{\omega}\,t}$ given by the frequency operator $\hat{\omega}$, acting on a constant complex start probability ***quantity*** $|\psi\rangle = |\psi(0)\rangle$, see (2.47), thus

(2.62) $|\psi(t)\rangle = e^{-i\hat{\omega}\,t}|\psi\rangle.$

From here rewritten (2.61)

(2.63) $i\frac{\partial}{\partial t}e^{-i\omega\cdot t}|\psi\rangle = \hat{\omega}e^{-i\omega\cdot t}|\psi\rangle$

When the operator $\hat{\omega}$ is a constant, $u_\omega(t) = e^{-i\hat{\omega}\,t}$ is a solution to the equation (2.61), i.e.

(2.64) $i\frac{\partial}{\partial t}e^{-i\omega t} = \hat{\omega}\,e^{-i\omega t}.$

By rewriting $\hat{H} = \hbar\hat{\omega}$ in equation (2.61) we have the famous parameter dependent

(2.65) **Schrödinger equation** $\boxed{i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle}$

2.2.4. Stationary Eigenstates and Eigenvalues

With the use of Dirac's ⟨bra|, |ket⟩ notation, e.g. about probability functions $\psi(y)$, as $|\psi\rangle$ vectors in a Hilbert space, $|\psi\rangle \in \mathcal{H} = L^2(\mathbb{R})$, we defined the scalar product $\langle\psi_1|\psi_2\rangle$.

We consider the operator $\hat{H}$ as one linear transformation (possibly a matrix) which act on $\mathcal{H}$.

[57] Attention, this is not a definition, but a possible assumed connection from physics to the cyclical oscillator idea.

[58] The example is also mentioned in § 1.3.4.3. In quantum physics in the 20th century, it has been a tradition of measuring ω in [eV]. You could instead use the more fundamental unit radians per second [Hz/2π] for energy by providing a better understanding of the relationship with frequency.

We will look at the set of eigenvectors $|\phi\rangle \in \mathcal{H}$, which are mutually orthogonal $|\phi_1\rangle \neq |\phi_2\rangle \Rightarrow \langle\phi_1|\phi_2\rangle = 0$ with associated scalar eigenvalues E_ϕ, which meet the condition

(2.66) $\hat{H}|\phi\rangle = E_\phi|\phi\rangle.$

If $\hat{H}$ is a Hermitian operator, its eigenvalues are real, $E_\phi \in \mathbb{R}$.

The interpretation of these eigenvalues is the ***quantum energies*** E_ϕ of a physical ***entity*** Ψ.

The Schrödinger equation for the circle oscillator is given by (2.61) and the stationary eigenvalue equation

(2.67) $\hat{H}_\omega|\psi(t)\rangle \doteq E_\omega\,|\psi(t)\rangle$

can be written in the form, $\hbar\hat{\omega}|\psi(t)\rangle \doteq E_\omega|\psi(t)\rangle$, and further

(2.68) $i\frac{\partial}{\partial t}e^{-i\hat{\omega}\,t}|\psi\rangle = \hat{\omega}\,e^{-i\hat{\omega}\,t}|\psi\rangle \doteq \frac{1}{\hbar}E_\omega\,e^{-i\hat{\omega}\,t}|\psi\rangle.$

We say from (2.62)-(2.64), that the circle oscillators have the continuous eigenvalue spectrum $\boxed{E_\omega = \hbar\omega}$ with eigenvector functions of the type $|\psi(t)\rangle = e^{-i\omega t}|\psi\rangle$.

We often put $\hbar = 1$ and omit $\hbar$, and simply apply $E_\omega = \omega$. See also (2.57).

2.2.5. Commutator Relations

Above in (2.56) the quantum mechanical commutator was associated by correspondence with the classical Poisson brackets, see (2.26)–(2.28)

(2.69) $-i\left[\hat{F}(t),\hat{H}\right] \leftrightarrow \{f,H\}$

Similarly, the Hamilton formalism for the canonical quantities q and p (2.34) we transfer to the commutator between the **canonical conjugate operator**s $\hat{q}$ and $\hat{p}$

(2.70) $[\hat{q},\hat{p}] = i\hbar \sim \{q,\ p\} = 1\,.$

Further on, we will apply $\hbar = 1$.[59]

As in the classical picture of an ***entity*** Ψ_Σ, the quantum mechanical picture can consist of a set of the **canonical conjugate operators** $\{\hat{q}_j\}$ and $\{\hat{p}_j\}$ for j=0,1, … , N

With inspiration from the classical Poisson brackets (2.32)–(2.34), from (2.70) we write

(2.71) $[\hat{q}_j,\hat{p}_k] = i\,\delta_{jk}\,, \quad [\hat{q}_j,\hat{q}_k] = 0\,, \quad [\hat{p}_j,\hat{p}_k] = 0 \qquad \text{for } j,k = 0,1,\dots,N$

These **commutator** relationships are fundamental to quantum mechanics.

Looking at an arbitrary operator $\hat{F}_{(\hat{q}_j,\hat{p}_k)}$, that does not depend explicitly on the parameter $t \in \vec{\mathbb{R}}$, so that $\frac{\partial\hat{F}}{\partial t} = 0$, we can write (2.53) as

(2.72) $\frac{d}{dt}\hat{F}_{(\hat{q}_j,\hat{p}_k)} = -i\left[\hat{F}_{(\hat{q}_j,\hat{p}_k)},\hat{\omega}\right] = i\left[\hat{\omega},\hat{F}_{(\hat{q}_j,\hat{p}_k)}\right].$

The assumption $\frac{\partial\hat{F}}{\partial t} = 0$ is reasonable because we prohibit $t \in \vec{\mathbb{R}}$ advancing causality.

If you anyway construct a $\hat{F}_{(\hat{q}_j,\hat{p}_k,t)}$ design, so that it is explicitly dependent on $t \in \vec{\mathbb{R}}$ and enables $\frac{\partial\hat{F}}{\partial t} \neq 0$, *the human mind has taken power over physics!* – Or at least power over the operator idea. We remember that t is constructed as a parameter, not a physical ***quantity***.

We should also remember that if we try to interpret t as time, time is transcendental for us, as an internal property of our minds (memory). You cannot know your own mind per se! only that it exists. Put more directly; t cannot be measured, since it is the measuring parameter itself. (see Chapter 1 – If you are lucky, you can synchronise t with a count of a ω_c clock.)

[59] $\hbar = 1$ can be achieved when the energy and the angular frequency have the same measure.