2.2.2. The Operator Quantised Circle Oscillator

We introduced in section 1.7.5 (1.60) a ***quantity*** for the **circle oscillator**

(2.37) $$q_\omega = q_\omega(t) = \tilde{q}_\omega u_\omega = \tilde{q}_\omega e^{-i\omega t}$$

and its information development parameter derivative as a development change ***quantity*** (1.61)

(2.38) $$\dot{q}_\omega = \frac{\partial}{\partial t} q_\omega(t) = -i\omega\cdot\tilde{q}_\omega e^{-i\omega\, t} = -i\omega\cdot q_\omega .$$

From this I introduced (1.69) the imaginary differential operator $i\frac{\partial}{\partial t}$ for a circle oscillator

(2.39) $$i\frac{\partial}{\partial t}(\varphi) = \omega\cdot(\varphi).$$

Now we can convert the multiplication operation $\omega\cdot$ to an operator $\widehat{\omega}$ and write

(2.40) $$\widehat{\omega}(\varphi) \coloneqq i\frac{\partial}{\partial t}(\varphi).$$ Some would prefer to write this $\frac{1}{\hbar}\widehat{H}(\varphi) \coloneqq i\frac{\partial}{\partial t}(\varphi)$.

We recall, that φ is an arbitrary abstract complex test function for the operator.

Making an alternative rewriting of the unitary operator concept: Where we in section 1.7.5 introduced $u_\omega = e^{-i\omega t}$ transformed into an operator $e^{-i\widehat{\omega}}(t') = u_{\widehat{\omega}}(t')$

(2.41) $$u_\omega(t) = e^{-i\omega t} \sim e^{-i\widehat{\omega}\, t'} = e^{-i\widehat{\omega}}(t') = u_{\widehat{\omega}}(t') \quad \text{or just} \quad u_\omega \sim u_{\widehat{\omega}}$$

– A funny way to promote understanding of the unitary one parameter unitary group elements $u_{\widehat{\omega}}(t')$
I rewrite the operator $\widehat{\omega}$ to a differential operator with (2.40)

$$u_{\widehat{\omega}}(t') = u_{\widehat{\omega}t'} \sim e^{-i\widehat{\omega}\, t'} = e^{\frac{\partial}{\partial t}t'} = e^{\frac{\partial}{\partial t}}(t').$$

Here is an interpretation problem hidden in the claim $|e^{-i\widehat{\omega}\, t'}| = 1$ and thus $|e^{\frac{\partial}{\partial t}t'}| = e^0 \neq |e^1|$;
I encourage the reader to try a geometrical interpretation? We will try to consider this later.
– Anyway, no causality in t or t'. That is why we always count $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$ for any ***entity*** Ψ_ω. –

We remember that $u_\omega(t) = e^{-i\omega t} \in \mathbb{C}$ is a complex number, which apply $u_\omega u_\omega^* = 1$.
This of cause also applies to unitary operators $u_{\widehat{\omega}} u_{\widehat{\omega}}^* = 1$.
We now see that the rotation $\phi \rightarrow e^{i\phi}$ of $U(1)$ is given by the one parameter $\phi = \omega t \sim \widehat{\omega} t'$.
Now the claim is that the ***quantity*** ω through an operator $\widehat{\omega}$ gives the rotary oscillation.
The relative synchronisation parameter t' is the ideal measure that must be based on a given clock oscillator providing the timing parameter t, which relatively determines the quantities of all the oscillators Ψ_ω locally within the physical ***entity*** Ψ.

2.2.2.2. The *Spectrum* of Oscillators

As we mathematically want a real continuous[53] spectrum of oscillators: it is necessary *locally* to count on one continuous evolution parameter t_c as synchronised relatively to a clock oscillator ω_c.
I call the operator in (2.41) $u_{\widehat{\omega}} = u_{\widehat{\omega}}(t') = e^{-i\widehat{\omega}}(t') = e^{-i\widehat{\omega}\, t'}$, for the evolution operator.[54]
The ***quantity*** from (1.60) is now written

(2.42) $$q_\omega(t) = \tilde{q}_\omega e^{-i\omega t} \sim e^{-i\widehat{\omega}\, t'}\tilde{q}_\omega = u_{\widehat{\omega}}\tilde{q}_\omega$$

This expression for parameter dependence applies to each frequency ω in the ***spectrum***.
Each frequency contributes with circle oscillations operation $u_{\widehat{\omega}}$ with the weight $\tilde{q}_\omega \sim \tilde{q}(\omega) \in \mathbb{C}$.
Integrated over the ***spectrum*** using the inverse Fourier operator, one gets the integrated parameter dependent ***quantity*** as follows

(2.43) $$q(t) = \int_{\mathbb{R}} d\omega\, e^{-i\omega\cdot t}\,\tilde{q}(\omega)$$

[53] The overall knowledge of quantum physics tells us this continuity is impossible. The finite quantum energy $\hbar\omega$ for each oscillator will cause an infinite energy density. The ancient atomic idea demanded that all ***entities*** in a physical universe will be countable. This tell us to judge, that the density of oscillators must be finite. Remember this for physics, but in the pure mathematical Fourier treatment of a spectrum this is ignored.

[54] The operand t may in some cases be replaced by other ***quantities***, e.g., the ***quantity*** q_0, more about this later.

Up till now, the treatment of these ***quantities*** for the oscillators has been abstract, and do not necessarily show directly observable magnitudes. We must look at the probability of observing, measuring, or determining a ***quantity***.

2.2.3. Quantised Probability

We will now use Dirac's notation ⟨bra|, |ket⟩ in connection with probability functions $\psi(y)$, that can be defined as vectors in a Hilbert space of square integrable functions. $\psi \in \mathcal{H} = L^2(\mathbb{R})$.
We can then define a scalar product

(2.44) $$\langle\psi_1|\psi_2\rangle \coloneqq \int \psi_1^*(y)\,\psi_2(y)dy$$

The integrand represents the probability distribution $p(y)dy = \psi^*(y)\psi(y)dy$ for $\psi(y)$.
Has the ***entity*** Ψ an observable ***quantity*** in physics, we will represent it by a Hermitian[55] ***quantisation*** operator $\hat{F}$, and let it work on an associated probability function ψ, as $(\hat{F}\psi)(y)$.
The expectation value[56] of the observable ***quantity*** of the operator $\hat{F}$ is

(2.45) $$\langle\hat{F}\rangle_\psi = \langle\psi|\hat{F}\psi\rangle = \langle\psi|\hat{F}|\psi\rangle = \int_{-\infty}^{\infty}\psi^*(y)\,\hat{F}\psi(y)dy, \qquad \text{where } |\psi\rangle \sim \psi \text{ and } \psi^* \sim \langle\psi|.$$

The parameter-dependent expectation value of $\hat{F}$ is written

(2.46) $$\langle\hat{F}\rangle_t = \langle\psi(t)|\hat{F}|\psi(t)\rangle,$$

wherein the development one parameter dependent probability function from $|\psi\rangle$ is given by

(2.47) $$|\psi(t)\rangle = e^{-i\widehat{\omega}t}|\psi(0)\rangle = e^{-i\widehat{\omega}t}|\psi\rangle$$

The expectation value (2.46) of $\hat{F}$ can now be written as

(2.48) $$\langle\hat{F}\rangle_t = \langle\psi|e^{i\widehat{\omega}t}\hat{F}\,e^{-i\widehat{\omega}t}|\psi\rangle$$

2.2.3.2. Heisenberg Picture

The parameter dependent operator becomes

(2.49) $$\hat{F}(t) = e^{i\widehat{\omega}t}\hat{F}e^{-i\widehat{\omega}t}$$

We write the total parameter derivative of $\hat{F}(t)$ as

(2.50) $$\begin{aligned}\frac{d}{dt}\hat{F}(t) &= \frac{d}{dt}\left(e^{i\widehat{\omega}t}\hat{F}\,e^{-i\widehat{\omega}t}\right) = i\widehat{\omega}e^{i\widehat{\omega}t}\hat{F}e^{-i\widehat{\omega}t} + e^{i\widehat{\omega}t}\left(\frac{\partial\hat{F}}{\partial t}\right)e^{-i\widehat{\omega}t} + e^{i\widehat{\omega}t}\hat{F}\cdot(-i\widehat{\omega})e^{-i\widehat{\omega}t}\\ &= ie^{i\widehat{\omega}t}\left(\widehat{\omega}\hat{F} - \hat{F}\widehat{\omega}\right)e^{-i\widehat{\omega}t} + e^{i\widehat{\omega}t}\left(\frac{\partial\hat{F}}{\partial t}\right)e^{-i\widehat{\omega}t} = i\left(\widehat{\omega}\hat{F}(t) - \hat{F}(t)\widehat{\omega}\right) + e^{i\widehat{\omega}t}\left(\frac{\partial\hat{F}}{\partial t}\right)e^{-i\widehat{\omega}t}.\end{aligned}$$

By introducing the commutator product of the general form

(2.51) $$[b,d] = b\,d - d\,b \quad \text{or} \quad [q,p] = q\,p - p\,q \quad \longrightarrow \quad [\hat{q},\hat{p}] = \hat{q}\,\hat{p} - \hat{p}\,\hat{q}$$

that is the operator-commutator, which is a relation expressed by the brackets [,]
by that, the derivative becomes

(2.52) $$\frac{d}{dt}\hat{F}(t) = -i\left[\hat{F}(t),\widehat{\omega}\right] + e^{i\widehat{\omega}t}\left(\frac{\partial\hat{F}}{\partial t}\right)e^{-i\widehat{\omega}t}.$$

This last term is an explicit parameter derivative contracted by the form (2.49) to the Heisenberg equation picture

(2.53) $$\frac{d}{dt}\hat{F}(t) = -i\left[\hat{F}(t),\widehat{\omega}\right] + \frac{\partial\hat{F}(t)}{\partial t} \quad \Longrightarrow \quad i\frac{d}{dt}\hat{F} = \left[\hat{F},\widehat{\omega}\right], \quad \text{maintain } \frac{\partial\hat{F}}{\partial t} = 0.$$

Comparing with the classical Hamilton formulation (2.26) with the Poisson bracket, we have

(2.54) $$\frac{d}{dt}f = \{f,H\} + \left(\frac{\partial f}{\partial t}\right).$$

we see an equivalence between the classical Poisson brackets and the operator-commutator

[55] For a Hermitian operator applies $\hat{F} = \hat{F}^{\mathrm{H}}$; for complex functions $f^*(y) = f(-y)$; and matrices conjugate transposed $\hat{F} = \hat{F}^{*T}$.

[56] or the mean measure of the ***quantity*** for a large ensemble of cases.