2.2. The Hamilton Function

The second order Euler-Lagrange equation (2.7) can be directly used when we include spatial aspects. Here we will go over to the Hamilton formalism, where, instead of the parameter derivative $\dot{q}_i$ of the generalised ***quantities*** q_i, we introduce the connected *momentum* ***quantities*** p_i,

(2.13) $$p_i \;=\; \frac{\partial L}{\partial \dot{q}_i}.$$

2.2.1.2. Generalised Canonical *Quantities*

q_i, p_i is now called the ***canonical quantities***, and p_i is called the ***conjugate quantities*** to q_i.
As with the ***quantity*** $q = \{q_0, \dots, q_N\}$ we use $p = \{p_0, \dots, p_N\}$ the combined ***quantities*** q and p for the sum ***entity*** Ψ_Σ, which consists of $N{+}1$ sub ***entities***, for $i = 0,1,\dots,N$,
and where we imply all linear relationships, e.g., $H(q,p) = \sum_i H(q_i,p_i)$. (remark no parameter t)
Traditionally the generalised set of (q,p) is called a point in a so called *phase-space*. (not natural space)
By inserting (2.13) in (2.7) we can note the classical concept of '**the force**'[50]

(2.14) $$\dot{p}_i \;=\; \frac{\partial L}{\partial q_i}$$

Using a Legandre transformation[51] we can switch the function dependency of the two independent variable argument ***quantities***, $(q,\dot{q}) \leftrightarrow (q,p)$. In that, we use the total differential of L from (2.3), and insert the definition (2.13) and (2.14), we get

(2.15) $$dL = \frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \left(\frac{\partial L}{\partial t}dt\right) \;=\; \dot{p}_i\,dq_i + p_i\,d\dot{q}_i + \left(\frac{\partial L}{\partial t}dt\right)$$
$$= \dot{p}_i\,dp_i + d(p_i\,\dot{q}_i) - \dot{q}_i\,dp_i + \left(\frac{\partial L}{\partial t}dt\right).$$

By moving $d(p_i\dot{q}_i)$ from the right to the left side and change the sign, we get

(2.16) $$d(p_i\,\dot{q}_i - L) \;=\; -\dot{p}_i\,dq_i + \dot{q}_i\,dp_i - \left(\frac{\partial L}{\partial t}dt\right)$$

Comparing the argument in the differential with the energy function (2.11),
and using (2.13), we now form the **Hamilton function**.

(2.17) $$H(q,p,t) = p\cdot\dot{q} - L(q,\dot{q},t).$$

This function formula changes the dependence of arguments between the ***quantities***:

(2.18) $$(q,p) \leftrightarrow (q,\dot{q})$$

We look at the differentials of the Hamilton function $H(q,p,\mathrm{t})$, and then compares with (2.16)

(2.19) $$dH \;=\; \frac{\partial H}{\partial q_i}dq_i \;+\; \frac{\partial H}{\partial p_i}dp_i \;+\left(\frac{\partial H}{\partial t}dt\right),$$

(2.20) $$dH \;=\; -\dot{p}_i\,dq_i \;+\; \dot{q}_i\,dp_i \;-\left(\frac{\partial L}{\partial t}dt\right).$$

Instead of the Euler-Lagrange equation (2.7) we form **Hamilton's canonical equations**

(2.21) $$\dot{q}_i \;=\; \frac{\partial H}{\partial p_i} \qquad \sim \qquad \frac{dq_i}{dt} \;=\; \frac{\partial H}{\partial p_i}$$

(2.22) $$\dot{p}_i \;=\; -\frac{\partial H}{\partial q_i} \qquad \sim \qquad \frac{dp_i}{dt} \;=\; -\frac{\partial H}{\partial q_i}$$

These **canonical equations** are the stationary condition for the physical ***entity*** Ψ to be stable with the ***quantities*** $q,\ p$ and $H(q,p)$

(2.23) We have the explicit parameter derivative $\dfrac{\partial H}{\partial t} \;=\; -\dfrac{\partial L}{\partial t} \quad (=0)$

The $(=0)$ is preferred in the model for the ***entity*** Ψ to make it external **conservative**.

[50] This ***quantity*** expresses the ***quality*** that Newton and classical physics interpret as the concept of **force**.
[51] The Legendre transformation $pdx = d(px) - xdp$ is connected to integration by parts $\int p\,dx = px - \int x\,dp$.

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2.2.1.3. The Poisson Bracket

Let us look at some function $f(q,p)$ that depends on the canonical ***quantities*** q and p.
When (2.21) and (2.22) applies, the total parameter differential of f becomes

(2.24) $$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i$$
$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial f}{\partial t} + \{f,H\}$$

Here For simplification of that term, we will use the *Poisson brackets* defined as follows

(2.25) $$\{f,g\} \;=\; \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$ (remember the implicit sum $\sum_i$ for double indices i)

This expresses the difference between the total and the explicit parameter derivative of f, hence

(2.26) $$\frac{df}{dt} \;=\; \{f,H\} + \left(\frac{\partial f}{\partial t}\right)$$

We prefer the indifferent $\frac{\partial f}{\partial t} = 0$ for all functions $f(q,p)$ without explicitly dependent of t, so

(2.27) $$\frac{df(q,p)}{dt} \;=\; \{f,H\}.$$

The condition that a ***quantity*** function f may be a conserved integral of all the movements and internal changes for the ***entity*** Ψ for all one parameter values $\forall t \in \vec{\mathbb{R}}$,[52] is that also the total development of one parameter derivative vanishes, $\frac{df}{dt} = 0$

(2.28) $$\{f,H\} \;=\; 0$$

Special for H we get

(2.29) $$\{H,H\} \;=\; 0$$

This is consistent with, that the Hamilton function $H(q,p)$ is development independent of one parameter t, when (2.21) and (2.22) apply. Written in detail as in (2.24), we have

(2.30) $$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial t} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = \{H,H\} = 0$$

We note now that the Hamilton function $H(q_i,p_i)$ must be designed so that q_i, p_i is explicitly independent, i.e., that

(2.31) $$\frac{\partial p_j}{\partial q_i} = 0 \quad \text{and} \quad \frac{\partial q_j}{\partial p_i} = 0 \quad \text{for all} \quad \forall i, \forall j \in \{0,1,2,\dots N\} \subset \mathbb{N}$$

Then the fundamental relationships we rewrite by Poisson brackets for **canonical** q,p

(2.32) $$\{q,q\} \;=\; 0 \quad \Leftarrow \quad \frac{\partial q_j}{\partial q_i}\frac{\partial q_j}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial q_j}{\partial q_i} = 1\cdot 0 - 0\cdot 1 = 0$$

(2.33) $$\{p,p\} \;=\; 0 \quad \Leftarrow \quad \frac{\partial p_j}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial p_j}{\partial p_i}\frac{\partial p_j}{\partial q_i} = 0\cdot 1 - 1\cdot 0 = 0$$ (remember $\sum_{ij}$)

(2.34) $$\{q,\ p\} \;=\; \sum \delta_{ij} \to N \quad \Leftarrow \quad \frac{\partial q_j}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial p_j}{\partial q_i} = (1\cdot 1 - 0\cdot 0 = 1)_{i=j} \to N = \textstyle\sum_{i=j}\delta_{ij}$$

Hamilton's canonical equations (2.21) and (2.22) are then written as

(2.35) $$\{q,H\} \;=\; \dot{q} \;=\; \frac{\partial H}{\partial p} \;=\; \sum_i \frac{\partial H}{\partial p_i}$$

(2.36) $$\{p,H\} \;=\; \dot{p} \;=\; -\frac{\partial H}{\partial q} \;=\; -\sum_i \frac{\partial H}{\partial q_i}$$

The advantage with this formulation is that we do not need an explicit external parameter t.
We will not in this book go further in to Liouville's Theorem etc. the literature is rich in this.

[52] In practice we are limited to $\forall t \in [t_A, t_B]$ from the beginning A to the end B of the ***entity*** Ψ.