2. The Parameter Dependent Mechanics

2.1. The Lagrange Formalism

2.1.1. The Lagrange Function

We will now look on a function representing the ability for change. I will call this ***quality*** "*the portable energy*"[48] for a physical ***entity*** Ψ and introduce the *Lagrange function*

(2.1) $$L(q,\dot{q})$$

based on the ***quantity*** q and the one parameter derivative ***quantity*** $\dot{q}$, as introduced in § 1.7.1.1 above. We have defined the three ***quantities*** (1.43)

(2.2) $$q_i(t),\ \dot{q}_i(t) \quad \text{and} \quad L\big(q_i(t),\dot{q}_i(t),t\big) \quad \text{as functions of the parameter } t \in \vec{\mathbb{R}}\,.$$

It is here decisive to specify, that the parameter t does not control the ***entity*** Ψ, but is only used to synchronise the measurements of all the oscillators, and there is no causality between the different oscillators, as indicated in the previous chapter.

With help of the Dirac delta function (1.83) section 1.7.7, we confirm that the parameter does t do not transfer causal dependence from $q(t)$ to $q(t')$ for $t' \neq t$, nor to the ***quantity*** $\dot{q}(t)$, and thence not to $L\big(q_i(t),\dot{q}_i(t),t\big)$. The Lagrange function $L(q,\dot{q})$ is only explicitly dependent on these two ***quantities***, the changing ***quantity*** q, and the of this independent change ***quantity*** $\dot{q}$.

As we saw for the oscillators (in section 1.7.5) the higher order derivative of these $\ddot{q}_\omega$, $\dddot{q}_\omega$, … just are real factorised repetitions of these same two ***quantities*** q_ω and $\dot{q}_\omega$.

These two q_ω and $\dot{q}_\omega$ are mutually orthogonal in an circle oscillator since they are separated by the complex factor $i\omega$, see (1.61). The Dirac delta function (1.88) also shows that q_ω and $q_{\omega'}$ are orthogonal for $\omega' \neq \omega$, and therefore independent of each other.

In the following, we can count on entities Ψ_Σ, that consist of $N{+}1$ subdivided entities Ψ_i for $i = 0,1,\dots,N$, so that the ***quantities*** $q = \{q_0,\dots,q_N\}$ and $\dot{q} = \{\dot{q}_0,\dots,\dot{q}_N\}$ is valid for Ψ_Σ in all linear relationships. With the ***quantities*** q and $\dot{q}$, we subsequent implicit understand possibilities of indices, e.g. $L(q,\dot{q}) = \sum_i L(q_i,\dot{q}_i)$.

We now look at the derivative of the Lagrange function $L(q,\dot{q},t) = \sum_i L(q_i,\dot{q}_i,t)$,

(2.3) $$dL = \frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt = \sum_i \frac{\partial L}{\partial q_i}dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt$$

Here we must take note of two notations.

- The one is the hard d in dx for the total differential of the variable x, while $\partial/\partial y$ is used as the partial derivative per the explicit variable y.
- The second is the summation implied over double indices (i) occurring in each part of the addition. In the first part of equation (2.3) the implied $\sum_i$ is omitted but shown in the last part.[49]

The ***quantities*** q_i are often called generalised coordinates. (Regarding space coordinates.)

Here I just will call q_i for the ***generalised quantities***, for a physical ***entity*** Ψ.

Referring to the definitions of q_i, $\dot{q}_i$ as dependent on the internal parameter t, as their arguments, we can also write $q_i(t), \dot{q}_i(t)$, and hence the parameter dependent Lagrange function as: $L(q_i(t),\dot{q}_i(t),\mathrm{t}) = L(q_i,\dot{q}_i,\mathrm{t})$

Seen from the external we will process t, q_i and $\dot{q}_i$ as three linearly independent inputs for the Lagrange function $L(q_i,\dot{q}_i,\mathrm{t})$ representing the ability of change for the ***entity*** Ψ.

[48] Is this energy negative it represents the binding of the ***entity*** Ψ to its surroundings, is it positive, it has some free kinematics.

[49] This reduced notation with an implicit sum over twice repeating indices was introduced by Einstein.

2.1.2. Action

We will now look at what happens from one parameter point t_{A} to one other t_{B}.
To determine the parameter points, we need to identify two events:
The first event A, then the second event B.
We determine by measurement $q_{\mathrm{A}}{=}q(t_{\mathrm{A}})$ and $q_{\mathrm{B}}{=}q(t_{\mathrm{B}})$. How much will happen from A to B?
We collect the *portable energy* $L(q,\dot{q},\mathrm{t})$ from A to B and get all the **action** by the integral.

(2.4) $$S = \int_{t_A}^{t_B} L(q,\ \dot{q},\ t)dt$$

The goal is to make this ***quantity*** stable and preferable minimised.
We do not necessarily know what $q(t)$ is, for $t_{\mathrm{A}} < t < t_{\mathrm{B}}$, where $t \in \vec{\mathbb{R}}$.
We think $q(t)$ as an arbitrary unknown function and add a minor variation $\delta q(t)$ to this.
Our guess function is the varying ***quantity*** $q(t){+}\delta q(t)$, where we shall maintain the endpoints A and B, so that the variations are fixed there $\delta q(t_{\mathrm{A}}) = \delta q(t_{\mathrm{B}}) = 0$. Then we look at the difference between these functions between A and B in the first order of δq.
The variation δS of S, when we go from $q(t)$ to $q(t){+}\delta q(t)$ for $\forall t \in]t_{\mathrm{A}},t_{\mathrm{B}}[$, will be

(2.5) $$\delta S = \int_{t_A}^{t_B} L(q{+}\delta q,\ \dot{q}{+}\delta\dot{q},\ t)dt - \int_{t_A}^{t_B} L(q,\ \dot{q},\ t)dt = \int_{t_A}^{t_B}\left(\delta q_i\frac{\partial L}{\partial q_i} + \delta\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right)dt \ \rightarrow 0.$$

We want $\delta S = 0$, as the action S herby may achieve a stable value, a desired minimum.
As $\delta\dot{q} = \frac{d\delta q}{dt}$ we get by shared integration of the last part second term in the bracket

(2.6) $$\delta S = \left[\delta q_i \cdot \frac{\partial L}{\partial \dot{q}_i}\right]_{t_A}^{t_B} + \int_{t_A}^{t_B} \delta q_i\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)dt = 0$$

The integrated first part disappears, because $\delta q(t_{\mathrm{A}}) = \delta q(t_{\mathrm{B}}) = 0$.
What remains is the **Euler-Lagrange** equation

(2.7) $$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

This equation is the condition that S is stable in an extremum (or minimum) $\delta S = 0$.

2.1.3. The Conservative Energy

We look at the total parameter derivative of the Lagrange function for the ***entity*** Ψ

(2.8) $$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial t} + \frac{\partial L}{\partial t}.$$

The first term in this addition is rewritten by definition $\dot{q}_i = \frac{\partial q_i}{\partial t}$ and using (2.7)

(2.9) $$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial t} + \frac{\partial L}{\partial t} = \dot{q}_i\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial t} + \frac{\partial L}{\partial t} = \frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial t},$$

and move the left to the opposite side of the equation

(2.10) $$\frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L\right) + \frac{\partial L}{\partial t} = 0$$

The term in the brackets is often called the energy function

(2.11) $$h(q,\dot{q}) \coloneqq \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L$$

Equation (2.10) may be considered as the total parameter derivative of h;

(2.12) $$\frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

The case the Lagrange function is not explicitly dependent on the parameter t, but only implicitly dependent through q and $\dot{q}$, as $L(q,\dot{q}) = \sum_i L(q_i,\dot{q}_i)$, gives $\frac{\partial L}{\partial t} = 0$.

This express, that (2.12) the energy function $h(q,\dot{q})$ is a preserved constant $\frac{dh}{dt} = 0$.

In the next section, we will instead of $h(q,\dot{q})$ introduce the Hamilton function $H(q,p)$.