1.7. The Cyclic Rotation

1.7.1.1. A *Entity* in Physics and its *Quantitative* Functions

Changes are the reason that we can experience the world. What we experience are ***entities***.
An ***entity*** Ψ in physics shall be experienced with some given ***primary qualities*** that must be allocated ***quantities*** q_i that are measurable.
To make the ***quantities*** for changing, we must write the changing ***quantities*** as a function $q_i(t)$ of a parameter – the development parameter $t \in \vec{\mathbb{R}}$, which we will choose as a countable measure as described above. A change in the ***quantities*** $q_i(t)$ themself, we express in new derived functions $\dot{q}_i = q_i'(t)$. The extra ***quantities*** $\dot{q}_i$ associated with the ***entity*** Ψ will demand extra ***primary qualities*** in the ***category*** that characterises the ***entity*** Ψ. (See § 1.7.2. below)
For an ***entity*** Ψ in physics, we construct a function $L(q_i, \dot{q}_i)$ called the *Lagrangian*, or the Lagrange function of the corresponding ***quantities*** q_i and $\dot{q}_i$ as arguments.
In section 1.7.5, we will demonstrate that higher order parameter derivatives have no explicit impact or control on $L(q_i, \dot{q}_i)$.
We already here claim a new concept:

The Lagrange function $L(q, \dot{q})$ represents the *portable energy* for the ***entity***.

- For L positive; Ψ has a degree of freedom, and
- For L negative; Ψ is bonded to the surroundings[32].

For the ***entity*** Ψ in physics we now have three sets of ***quantities***:

(1.43) $$q_i(t),\ \dot{q}_i(t) \quad \text{and} \quad L\big(q_i(t), \dot{q}_i(t), t\big)\ .$$

From the way, they are defined, they are jointly associated with the parameter t. Since the parameter does not provide or add any external ***causal quality***. We claim now that the parameter t is internal to the physical ***entity*** Ψ. Seen from the outside of the ***entity*** Ψ, we will process q and $\dot{q}$ as linear independent ***quantities***. About this claim see section 2.1.1.

1.7.2. The Derivative Function

The change $f'(x)$ of function $f(x)$ can be defined as the differential quotient

(1.44) $$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

This is often short written as

(1.45) $$\frac{d}{dx} f(x) = \frac{df(x)}{dx} = f'(x).$$

Here, we consider the symbol $\frac{d}{dx}$ as an operator, who acts on the following part.[33]
Rules for the differentiation of functions can be found in the literature. Just to be mentioned here:

(1.46) $$\frac{d}{dx} y^{ax} = a\, y^x,$$

and the derivative of the exponential function is self-identical and follows this simple rule:

(1.47) $$\frac{d}{dx}\exp(x) = \exp(x) \quad \Leftrightarrow \quad \frac{d}{dx} e^x = e^x$$

Which enables the power series (1.26) as the Taylor series for the exponential function.

[32] The outstanding question here is: **what are the surroundings**? We need to consider that later.
[33] Used on a (), it $\frac{d}{dx}(\_)$ is applied on what is inside the brackets. Like the term $\frac{d(\_)}{dx}$

1.7.3. The Parameter Derived Quantity

For a ***quantity*** q as the function $q(t)$ of the parameter $t \in \vec{\mathbb{R}}$ we can write the parameter-derived ***quantity*** as

(1.48) $$\dot{q} = \frac{dq}{dt} = \frac{dq(t)}{dt} = \frac{d}{dt} q(t) = q'(t),$$

and once again the parameter derived from this

(1.49) $$\ddot{q} = \frac{d^2 q}{d^2 t} = \frac{d^2 q(t)}{d^2 t} = \frac{d}{dt}\left(\frac{d}{dt} q(t)\right) = \frac{d}{dt} q'(t) = q''(t)$$

When we use the terms $\dot{q}$ and $\ddot{q}$ it is implicitly given, that it is the derivative concerning the information development parameter $t \in \vec{\mathbb{R}}$ for the ***quantity*** q.[34]
Specific expressed, the differentiated $\dot{q} = q'(t)$ and the twice differentiated $\ddot{q} = q''(t)$.

1.7.4. The Circular Rotation and the Unitary Group $U(1)$

The circular rotation has a ***primary quality*** given by the Euler circle $e^{i\phi} \in \mathbb{C}$, $\phi \in \mathbb{R}$.
The real ***quantity*** ϕ indicates the angle of rotation in the movement or just the difference in the rotation angle around the circle. The ***quantity*** ϕ is often called the chronometric phase angle.
This ***primary quality*** is represented by the circle group, which is the multiplicative group of complex numbers with absolute value 1, the complex number set

(1.50) $$\mathbb{T} = \{u \in \mathbb{C} \mid |u| = 1\}.$$

The circle group is synonymous with the **unitary group** $U(1)$ consisting of the exponential map

(1.51) $$\mathbb{R} \to \mathbb{T} :\ \phi \to u \qquad \text{where} \quad u = e^{i\phi} \ = \ \cos\phi + i \sin\phi\ .$$

The following calculation rule applies to group elements in $\mathbb{T}$: $e^{i\phi_1} e^{i\phi_2} = e^{i(\phi_1 + \phi_2)}$.
Setting the complex number $u = e^{-i\phi}$ and thus the complex conjugate $u^* = e^{i\phi}$, we get

(1.52) $$u^2 = |u|^2 = u^* u = \ e^{i\phi} e^{-i\phi} = \ e^{i(\phi - \phi)} = \ e^0 = 1 \quad \in \mathbb{T}\,,$$

confirming that

(1.53) $$|u| = 1 \quad \text{for } \forall u \in \mathbb{T}$$

We note that the group homomorphism $\exp: \mathbb{R} \to \mathbb{T}$ linking the additive group $\mathbb{R}$ to the multiplicative group $\mathbb{T}$.
Looking at the diverse groups with isomorphic properties we have $\mathbb{T} \sim U(1) \sim SO(2) \sim \mathbb{R}/\mathbb{Z}$.
$SO(2)$ is the special orthogonal rotation group of real $2{\times}2$ matrices of the type

(1.54) $$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \qquad \text{with the determinant} \qquad \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} = 1$$

$\mathbb{R}/\mathbb{Z}$ represents the quotient group, is periodic identical, especially we see the formulation $\frac{\mathbb{R}}{2\pi}/\mathbb{Z}$ has the same periodicity as the circle group and $U(1)$. (See Figure 1.1)

The difference between the circle group and $U(1)$ is that the elements of the former are defined in the set by $\{u \in \mathbb{C} \mid |u|{=}1\}$, and the other the elements are $1{\times}1$ unitary matrices $[u]$, in the set $\{\,[u] \mid u{\in}\mathbb{C} \ \wedge\ u^* u{=}1\,\}$ corresponding to the complex numbers of the type $u = e^{i\phi} \in \mathbb{C}$, for $\phi \in \mathbb{R}$. – The unitary group $U(1)$ apply $u{\sim}[u]_{1\times 1}$.
The map $\mathbb{R} \to \mathbb{T} \subset \mathbb{C}: \phi \to u(\phi) = e^{-i\phi}$ is called for the *exponential map* of the real numbers to the unit circle in the complex plane.
This mapping is surjective but not injective. There is no causality in the real input $\phi \in \mathbb{R}$.

[34] This implicit notation $\dot{q}$ is a tradition dating from Newton, in contrast to the more detailed Leibniz notation $\frac{dq}{dt} = \frac{dq(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \lim_{\Delta t \to 0} \frac{q(t+\Delta t) - q(t)}{\Delta t}$, while the notation $q' = q_t' = q'(t) = \frac{dq(t)}{dt}$ is inherited from Lagrange.