(1.15) $t_m = m/f \approx t_n = n \in \vec{\mathbb{N}}$.

If you want a finer time measurement you must construct a faster cyclic clock with a higher frequency.[23]

We are now taking the cyclical circle clock $\odot_c^n$, with clock frequency $f_c$=1, and introduce a continuous real information development parameter $t \in \vec{\mathbb{R}}_+$ synchronised by the map (1.12)

(1.16) $TIMING\colon\ t \to t_n := \lfloor f_c{\cdot}t \rfloor = \lfloor t \rfloor$,

which synchronises the development parameter $t \in \vec{\mathbb{R}}_+$ with the timing $t_n \in \vec{\mathbb{N}}$, so that $\odot_c^{f_c\cdot t} \sim \odot_c^n$ is representing a continuous oscillating clock, in a circular motion with its own timing reference, as an autonomy clock, so that $f_c$=1 by definition.

Looking at the other circular motion $\odot^{ft} \sim \odot^m$, we can measure its oscillations[24] with the development parameter $t \in \vec{\mathbb{R}}_+$ from the timing clock $\odot_c^t \sim \odot_c^n$, where we prerequisite, that the relative proportion between $\odot^{ft} \sim \odot^m$ and $\odot_c^t \sim \odot_c^n$ is constant, as $f = m/n$ is constant, for a synchronous measure of the counts $m$ and $n$.

From the clock $\odot_c^t$ development parameter $t \in \vec{\mathbb{R}}_+$ we can calculate the other $\odot^{ft}$ oscillation time $T = 1/f = t_m/m$ for its circular motion and its own timing map will be

(1.17) $TIMING\colon\ t \to t_m = m{\cdot}T = m/f = \lfloor f{\cdot}t \rfloor{\cdot}T$ , with reference to the clock $\odot_c^t$ .

The number $m = \lfloor f{\cdot}t \rfloor$ counts laps in $\odot^m \sim \odot^{ft}$. Since we know that an angular turn in a circle is 2π, we introduce an autonomous and continuous number $\theta = 2\pi{\cdot}f{\cdot}t$ called for the phase angle of the circular oscillation. This phase angle $\theta \in \mathbb{R}$ then expressed an autonomous information development parameter for a circle of oscillation $\odot^{\theta/2\pi}$.

(1.18) $TIMING\colon\ \theta \to \lfloor \theta/2\pi \rfloor$, the autonomous phase angle timing is modulo $2\pi$.

The phase angle concept is widely used in quantum mechanics, optics, and electronics which we describe later below.

In (1.17) the frequency $f$ and oscillation time $T$ are measured relative to the clock $\odot_c^n$ just as the development parameter $t \in \vec{\mathbb{R}}_+$. It is common to use the same development parameter $t \in \vec{\mathbb{R}}_+$ for all cyclic conditions for any ***entity*** in local physics, with a constant relative relation to the clock $\odot_c^t$ .

Mathematically all these numbers are now accounted for as real ***quantities***: $\theta,\ t,\ f,\ \mathrm{T} \in \mathbb{R}$, although when measured, they can only provide by relatively counting the whole integer number, i.e., in rational number relationships.

Object examples of cyclic rotating watches:

- 1 oscillation in a Cs133 atom clock $\odot_{Cs}^1$
- 1 second with caesium clock $\odot_s^1 \sim \odot_{Cs}^{9192631770}$
- 1 minute 60 seconds $\odot_m^1 \sim \odot_s^{60}$
- 1 hour by 3600 seconds $\odot_h^1 \sim \odot_s^{3600}$
- 1 day with 86,400 seconds $\odot_d^1 \sim \odot_s^{86400}$
- 1 year of 31,556,926 seconds $\odot_y^1 \sim \odot_s^{31556926}$

The cyclic time can be represented by a circle of rotation.
The most elegant way in mathematics of representing a circle of rotation is Euler's circle formula, using complex numbers.

[23] A graduated scale for a hand pointer on a dial is a spatial measure of angle or distance, that does not indicate a true time measurement.
[24] As an association with the traditional continuous complex number clock oscillator, we can write $\odot^{ft} \equiv e^{-i2\pi ft} \equiv e^{-i\omega t}$, see below.









**1.4.1.2. The Circle Plan**[25]

The circle is a figure in the plane of Euclidean geometry. The Euclidean plane may traditionally be represented by $\mathbb{R}^2$, so that a point is represented by $(x_1, x_2) \in \mathbb{R}^2$ from origo (0,0).

The same point can be represented in the complex plane concept by $z \in \mathbb{C}$ from origo $0 \in \mathbb{C}$.

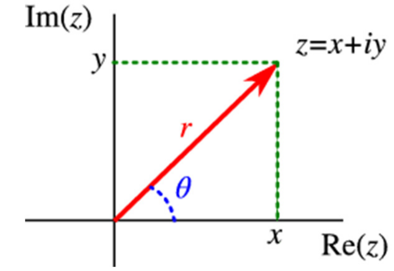
We connect the two representations $\mathbb{C}{\sim}\mathbb{R}^2$ of the Euclidean plane, as $z \sim (x_1, x_2)$ by the definition $z = x_1 + i\,x_2 = \operatorname{Re} z + i \operatorname{Im} z$, thus $(x_1, x_2) = (\operatorname{Re} z, \operatorname{Im} z) \in \mathbb{R}^2$

From these, the circle is defined by Euler's circle formula from the polar real representation

(1.19) $(r, \theta) \in \mathbb{R}^2 \longrightarrow z = r\cos\theta + i\,r\sin\theta \in \mathbb{C}$, with $(x_1, x_2) = (r\cos\theta,\ r\sin\theta) \in \mathbb{R}^2$

Which makes a complex number:

## 1.5. The Complex Numbers




The imaginary unit $i$ of the complex numbers is defined as the number that multiplied by itself gives $-1$

(1.20) $i{\cdot}i = i^2 - 1 \quad\Rightarrow\quad i = \sqrt{-1}$.

The complex numbers $z = x + iy \in \mathbb{C}$ are composed of one real part $\operatorname{Re}(z) = x$ and one imaginary part $\operatorname{Im}(z) = y$

(1.21) $z = \operatorname{Re}(z) + i\operatorname{Im}(z) = x + iy \leftrightarrow (x, y)$,

in which $x, y \in \mathbb{R}$, and $z \in \mathbb{C} \;\Rightarrow\; (x, y) \in \mathbb{R}^2 \;\leftrightarrow\; \mathbb{C}$,
$z$ is represented by a point $(x, y)$ in the plane.[26] A 'vector'[27] can point out the point

(1.22) $\vec{r} = \binom{x}{y} = (x, y)^T$, from an origo (0,0).

(1.23) $r^2 = x^2 + y^2 = x^2 - (iy)^2 = (x + iy)(x - iy) = z{\cdot}z^* \in \mathbb{R}$, where $z^* = x - iy$

$z^*$ is called for the complex conjugated to $z$ .

The complex numbers have the magnitude $|z| = |\vec{r}| = \sqrt{x^2 + y^2} \in \mathbb{R}$

In addition, we often use an absolute square on a complex number

(1.24) $|z|^2 = z{\cdot}z^*$ compared to the traditional form $|z|^2 = |\vec{r}|^2 = x^2 + y^2$

Here we notice the particularly significant difference in the notation used here

(1.25) $|i|^2 = i{\cdot}i^* = 1$ , while $i^2 = (i)^2 = i{\cdot}i = -1$

### 1.5.2. The Complex Exponential Function

To prepare the mathematical significance of the circle rotation we will use the complex exponential, which by definition is given by the power series:

(1.26) $e^z := 1 + \frac{z}{1!} + \frac{(z)^2}{2!} + \frac{(z)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(z)^n}{n!}$

[25] Much more about the circular rotation in the geometric plane concept later in chapter II. 5 below for the Geometric Algebra.
[26] The complex plane is represented by the Cartesian coordinate system with basis vectors $\hat{x}\perp\hat{y}$ and coordinate axes which are mutually perpendicular $x_{axis}\perp y_{axis}$ to make a point $(x, y)$ of the plane. The imaginary unit $i$ represent together with the real unit 1 the two perpendicular unit-basis-vectors. $\hat{\imath} = \hat{y} \perp \hat{1} = \hat{x}$ . More below § 4.1.3 and (5.142).
A point (x, y) in the abstract complex plane $z \in \mathbb{C}$ shall not be interpreted as existing in the natural physical 3D space.
[27] A traditional *radius vector* $\vec{r}$ can 'lift' an origo point (0,0) to an arbitrary point in the plane $(x, y)$. Or said in another way designate the point in the plane from origo.